

# Electromagnetic characteristics and effective gauge theory of double-layer quantum Hall systems

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The electromagnetic characteristics of double-layer quantum Hall systems are studied, with projection to the lowest Landau level taken into account and intra-Landau-level collective excitations treated in the single-mode approximation. It is pointed out that dipole-active excitations, both elementary and collective, govern the long-wavelength features of quantum Hall systems. In particular, the presence of the dipole-active interlayer out-of-phase collective excitations, inherent to double-layer systems, modifies the leading  $O(\mathbf{k})$  and  $O(\mathbf{k}^2)$  long-wavelength characteristics (i.e., the transport properties and characteristic scale) of the double-layer quantum Hall states substantially. We apply bosonization techniques and construct from such electromagnetic characteristics an effective theory, which consists of three vector fields representing the three dipole-active modes, one interlayer collective mode and two inter-Landau-level cyclotron modes. This effective theory properly incorporates the spectrum of collective excitations on the right scale of the Coulomb energy and, in addition, accommodates the favorable transport properties of the standard Chern-Simons theories.

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## I. INTRODUCTION

Laughlin's variational wave functions<sup>1</sup> gave impetus<sup>2</sup> to descriptions of the fractional quantum Hall effect<sup>3,4</sup> (FQHE) in terms of electron-flux composites. The Chern-Simons theories realize the composite-boson<sup>5-7</sup> and composite-fermion<sup>8-10</sup> pictures of the FQHE and have been successful in describing various aspects of the fractional quantum Hall (FQH) states.

An alternative approach,<sup>11</sup> that makes use of projection to Landau levels and bosonization as basic tools, has recently been developed to study the long-wavelength features of single-layer quantum Hall systems. There an effective vector-field theory is constructed, via bosonization, from the electromagnetic response of incompressible states. It refers to neither the composite-boson nor composite-fermion picture, but properly reproduces the results of the Chern-Simons (CS) theories at long wavelengths, thus revealing the universal long-wavelength characteristics of the incompressible FQH states.

For double-layer systems the quantum Hall effect exhibits a rich variety of patterns and physics, as observed experimentally<sup>12</sup> and discussed theoretically.<sup>13-22</sup> The Chern-Simons theoretic approaches, however, encounter some subtle problem, when generalized to double-layer

systems. They differ significantly<sup>18</sup> in the collective-excitation spectrum from a general magneto-roton theory of collective excitations based on the single-mode approximation (SMA), developed by Girvin, MacDonald and Platzman.<sup>23</sup>

The purpose of this paper is to extend the previous (projection + bosonization) approach to double-layer quantum Hall systems and to clarify the relation between the microscopic SMA theory and the CS theories. Double-layer systems generally support dipole-active intra-Landau-level collective modes,<sup>16,18</sup> in sharp contrast to single-layer systems where only the cyclotron mode remains dipole active.<sup>23</sup> [A dipole-active mode is the one whose spectral weight behaves like  $O(\mathbf{k}^2)$  in the long-wavelength limit  $\mathbf{k} \rightarrow 0$ . It is, as such, sensitive to long-wavelength probes.] We study the electromagnetic response of a double-layer system in the absence of interlayer tunneling, with the collective excitations taken into account in the SMA. It is shown that the presence of the dipole-active interlayer collective excitations modifies the  $O(\mathbf{k})$  and  $O(\mathbf{k}^2)$  long-wavelength response (i.e., the transport properties and characteristic scale) of the system substantially. The effective theory of the FQHE, reconstructed from this response, consists of three vector fields that represent the three dipole-active modes residing in the system. It properly embodies the SMA spectrum of collective excitations, along with the favorable transport properties of the CS theories.

In Sec. II we study the electromagnetic response of a single-layer Hall-electron system and present a general formalism for achieving projection onto the true Landau levels by a suitable unitary or  $W_\infty$  transformation. In Sec. III we examine the electromagnetic characteristics of a double-layer system and, in Sec. IV, construct an effective gauge theory of the FQHE. Section V is devoted to a summary and discussion.

## II. ELECTROMAGNETIC RESPONSE OF HALL ELECTRONS

Consider electrons confined to a plane with a perpendicular magnetic field  $B_z = B > 0$ . Our task in this section is to study how they respond to weak external potentials  $A_\mu(\mathbf{x}, t) = (A_0(\mathbf{x}, t), \mathbf{A}(\mathbf{x}, t))$ , with the action

$$S_1 = \int dt d^2\mathbf{x} \psi^\dagger(\mathbf{x}, t) (i\partial_t - \mathcal{H}) \psi(\mathbf{x}, t), \quad (2.1)$$

$$\mathcal{H} = \frac{1}{2M} (\mathbf{p} + e\mathbf{A}^B(\mathbf{x}) + e\mathbf{A}(\mathbf{x}, t))^2 + eA_0(\mathbf{x}, t), \quad (2.2)$$

where  $\mathbf{A}^B = \frac{1}{2}B(-y, 0)$  supplies a uniform magnetic field  $B$ . The Coulomb interaction will be included later. [We suppose that  $\mu$  runs over  $(t, x, y)$  or  $(0, 1, 2)$ , and denote  $\mathbf{A} = (A_1, A_2)$ , etc.] For conciseness we shall write  $x = (t, \mathbf{x})$ ,  $d^3x = dt d^2\mathbf{x}$ ,  $A_\mu(x) = A_\mu(\mathbf{x}, t)$ , etc., when no confusion arises; the electric charge  $e > 0$  is also suppressed by rescaling  $eA_\mu \rightarrow A_\mu$  in what follows.

The eigenstates of a free orbiting electron are Landau levels  $|N\rangle = |n, y_0\rangle$  of energy  $\omega_c(n + \frac{1}{2})$ , labeled by integers  $n = 0, 1, 2, \dots$ , and  $y_0 = \ell^2 p_x$ , where  $\omega_c \equiv eB/M$  and  $\ell \equiv 1/\sqrt{eB}$ . This level structure is modified in the presence of  $A_\mu(x)$ , and the effect of level mixing it causes is calculated by diagonalizing the Hamiltonian with respect to the true Landau levels  $\{n\}$ .

The actual calculation is best done in  $N$  space, where the Hamiltonian  $\langle n, y_0 | \mathcal{H} | n', y'_0 \rangle = \langle y_0 | \hat{\mathcal{H}}_{nn'} | y'_0 \rangle$  becomes an infinite-dimensional matrix in level indices and an operator in  $y_0$  and its conjugate  $x_0 \equiv i\ell^2 \partial / \partial y_0$ ;  $\mathbf{r} = (x_0, y_0)$  represents the center coordinate of an orbiting electron with uncertainty  $[r_1, r_2] = i\ell^2$ . With a unitary transformation  $\psi(\mathbf{x}, t) = \sum_N \langle \mathbf{x} | N \rangle \psi_N(y_0, t)$ , the one-body action is rewritten as<sup>24</sup>

$$S_1 = \int dt dy_0 \sum_{m,n=0}^{\infty} \psi_m^\dagger(y_0, t) \left( i\delta_{mn} \partial_t - \hat{\mathcal{H}}_{mn} \right) \psi_n(y_0, t),$$

$$\hat{\mathcal{H}} = \omega_c(Z^\dagger Z + \frac{1}{2}) + U(\hat{\mathbf{x}}, t), \quad (2.3)$$

where

$$U(x) = i\ell\omega_c \{Z^\dagger A(x) - A^\dagger(x)Z\} + \ell^2\omega_c A^\dagger(x)A(x) + A_0(x) + \frac{1}{2M} A_{12}(x) \quad (2.4)$$

summarizes the electromagnetic coupling and

$$A = (A_2 + iA_1)/\sqrt{2}, \quad A^\dagger = (A_2 - iA_1)/\sqrt{2}; \quad (2.5)$$

$A_{12} = \partial_1 A_2 - \partial_2 A_1$ . Note that in  $N$  space the coordinate operators  $\mathbf{x} = (x_1, x_2)$  take the matrix form

$$(\hat{x}_i)_{nn'} = (X_i)_{nn'} + r_i \delta_{nn'}, \quad (2.6)$$

where  $X_i$  are the hermitian matrices of a harmonic oscillator with  $[X_1, X_2] = -i\ell^2$ ; in the above, we have defined  $Z = (X_2 + iX_1)/(\sqrt{2}\ell)$  so that  $Z_{mn} = \sqrt{n} \delta_{m,n-1}$  and  $[Z, Z^\dagger] = 1$ . In this section we use the Landau gauge but all the manipulations are carried over to the symmetric gauge<sup>23,25</sup> as well.

The one-body action  $S_1$  is made diagonal in level indices by a suitable  $U(\infty)$  or  $W_\infty$  transformation  $G$  of the form

$$\psi_m^G(y_0, t) = \sum_{n=0}^{\infty} G_{mn}(x_0, y_0, t) \psi_n(y_0, t), \quad (2.7)$$

under which  $U$  undergoes the transformation

$$U^G = GUG^{-1} + \omega_c[G, Z^\dagger Z]G^{-1} - iG\partial_t G^{-1}. \quad (2.8)$$

A general program of projection onto the true Landau levels along this line was laid out earlier.<sup>24</sup> Here we complete it by presenting an  $O(U^2)$  expression, exact to all powers of derivatives, and how to handle the Coulomb interaction.

For diagonalization let us expand  $U$  in the  $U(\infty)$  basis  $\{\Gamma_{su} \equiv (Z^\dagger)^s Z^u / (s!u!)\}$ ,

$$U(\hat{\mathbf{x}}, t) = \sum_{s,u=0}^{\infty} U_{su}(\mathbf{r}, t) \Gamma_{su}, \quad (2.9)$$

where  $\mathbf{r} = (r_1, r_2)$ . Likewise, we write  $G = \exp[i\ell\eta]$  and  $\eta = \sum_{s,u=0}^{\infty} \eta_{su}(\mathbf{r}, t) \Gamma_{su}$ . With the choice

$$i\eta_{su} = [(s-u)\omega_c - i\partial_t]^{-1} U_{su} \quad (s \neq u), \quad (2.10)$$

$U^G = \sum_{s,u=0}^{\infty} (U^G)_{su} \Gamma_{su}$  is made diagonal to  $O(U^2)$ . In particular, for the lowest Landau level

$$(U^G)_{00} = U_{00} - \sum_{s=1}^{\infty} \frac{1}{s!} U_{0s} \frac{1}{s\omega_c - i\partial_t} U_{s0} + \dots, \quad (2.11)$$

apart from an (unimportant) total divergence  $\propto \partial_t(\dots)$ . In what follows we focus on the lowest Landau level (of practical interest) under a strong magnetic field. For conciseness we set the magnetic length  $\ell \rightarrow 1$  below.

Let us write out  $U_{su}$ . To this end define the field  $A(\hat{\mathbf{x}}, t)$  through the Fourier transform  $A(\hat{\mathbf{x}}, t) = \sum_{\mathbf{p}} A[\mathbf{p}, t] e^{i\mathbf{p} \cdot \hat{\mathbf{x}}}$  and normal-order it with respect to  $Z^\dagger$  and  $Z$ ,

$$A(\hat{\mathbf{x}}, t) = \sum_{\mathbf{p}} A[\mathbf{p}, t] F(\mathbf{p}) e^{-\frac{1}{4}\mathbf{p}^2} e^{i\mathbf{p} \cdot \mathbf{r}}, \quad (2.12)$$

$$F(\mathbf{p}) \equiv : e^{i\mathbf{p} \cdot \mathbf{X}} : = e^{\frac{i}{\sqrt{2}} p Z^\dagger} e^{\frac{i}{\sqrt{2}} p^\dagger Z}, \quad (2.13)$$

where  $p = p_2 + ip_1$  and  $p^\dagger = p_2 - ip_1$ . One can thus write

$$A(\hat{\mathbf{x}}, t) = \sum_{s,u=0}^{\infty} \Gamma_{su} \bar{\partial}^s \partial^u e^{\frac{1}{2}\bar{\partial}\partial} A(\mathbf{r}, t), \quad (2.14)$$

with  $\bar{\partial} = \frac{1}{\sqrt{2}}(\partial_{r_2} + i\partial_{r_1})$  and  $\partial = \frac{1}{\sqrt{2}}(\partial_{r_2} - i\partial_{r_1})$  acting on  $A(\mathbf{r}, t)$ . One may regard  $\bar{\partial}^s \partial^u e^{\frac{1}{2}\bar{\partial}\partial} A(\mathbf{r}, t)$  as representing the Fourier transform  $(ip/\sqrt{2})^s (ip^\dagger/\sqrt{2})^u e^{-\frac{1}{4}\mathbf{p}^2} A[\mathbf{p}, t]$ . Substitution of Eq. (2.14) into  $U$  yields

$$U_{00} = A_0^{[r]} + \frac{1}{2M} A_{12}^{[r]} + \omega_c(A^\dagger A)^{[r]} \equiv \chi^{[r]},$$

$$U_{s0} = \bar{\partial}^{s-1} (is\omega_c A^{[r]} + \bar{\partial}\chi^{[r]}) \quad (s \geq 1),$$

$$U_{0s} = \partial^{s-1} (-is\omega_c \bar{A}^{[r]} + \partial\chi^{[r]}), \quad (2.15)$$

where  $A^{[r]} \equiv e^{\frac{1}{2}\bar{\partial}\partial} A(\mathbf{r}, t)$ ,  $A_{12}^{[r]} \equiv e^{\frac{1}{2}\bar{\partial}\partial} A_{12}(\mathbf{r}, t)$ , etc.

Some care is needed in handling operators involving  $\mathbf{r}$  with  $[r_1, r_2] = i$ . Note that  $T_{\mathbf{p}} = e^{-\frac{1}{4}\mathbf{p}^2} e^{i\mathbf{p} \cdot \mathbf{r}}$  obeys the multiplication law

$$T_{\mathbf{p}} T_{\mathbf{k}} = e^{\frac{1}{2}\mathbf{p}^\dagger \mathbf{k}} T_{\mathbf{p}+\mathbf{k}} = e^{\frac{1}{2}(\mathbf{p} \cdot \mathbf{k} - i\mathbf{p} \times \mathbf{k})} T_{\mathbf{p}+\mathbf{k}}, \quad (2.16)$$

where  $\mathbf{p} \times \mathbf{k} \equiv \epsilon^{ij} p_i k_j = p_1 k_2 - p_2 k_1$  with  $\epsilon^{12} = 1$ . One can therefore write an operator product  $A^{[r]} B^{[r]}$  as

$$A^{[r]} B^{[r]} = \sum_{\mathbf{p}} T_{\mathbf{p}} (A * B)_{\mathbf{p}}, \quad (2.17)$$

$$(A * B)_{\mathbf{p}} = \sum_{\mathbf{k}} e^{\frac{1}{2}(\mathbf{p}^\dagger - \mathbf{k}^\dagger) \cdot \mathbf{k}} A[\mathbf{p} - \mathbf{k}, t] B[\mathbf{k}, t]. \quad (2.18)$$

Thus, back in ordinary (commuting)  $\mathbf{x}$  space,

$$\begin{aligned} A * B &= e^{-\partial \bar{\partial}'} A(\mathbf{x}) B(\mathbf{x}')|_{\mathbf{x}' \rightarrow \mathbf{x}} \\ &= A(\mathbf{x}) B(\mathbf{x}) - \partial A(\mathbf{x}) \bar{\partial} B(\mathbf{x}) + \dots \end{aligned} \quad (2.19)$$

With this rule one can translate operator products into ordinary functions and go through the calculation. Let us write  $U_{00}^G(\mathbf{r}, t) = \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{r}} e^{-\frac{1}{4}\mathbf{p}^2} \mathcal{U}_{00}^G[\mathbf{p}, t]$  with  $\mathcal{U}_{00}^G[\mathbf{p}, t] = A_0[\mathbf{p}, t] + \frac{1}{2M} A_{12}[\mathbf{p}, t] + \mathcal{U}_2[\mathbf{p}, t]$ . The  $\mathcal{U}_2[\mathbf{p}, t]$  denotes the contribution quadratic in  $A_\mu$ . We here quote only the  $\mathbf{p} = 0$  component  $\mathcal{U}_2[\mathbf{p} = 0, t] = \int d^2 \mathbf{x} \mathcal{U}_2(x)$  with

$$\begin{aligned} \mathcal{U}_2(x) &= -\frac{1}{2\omega_c} \left( A_{k0} + \frac{1}{2M} \partial_k A_{12} \right) D \left( A_{k0} + \frac{1}{2M} \partial_k A_{12} \right) \\ &\quad + \frac{1}{2} A_\mu D' \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{1}{2M} A_{12} D' A_{12} \end{aligned} \quad (2.20)$$

in compact notation. Here the correlation functions  $D = \sum_{n=1}^\infty D_n$  and  $D' = \sum_{n=1}^\infty n D_n$  are written in terms of

$$D_n = \frac{1}{(n-1)!} \frac{\omega_c^2}{(n\omega_c)^2 + \partial_t^2} e^{\frac{1}{2}\nabla^2} (-\frac{1}{2}\nabla^2)^{n-1}, \quad (2.21)$$

with  $\nabla^2 \equiv \partial_k \partial_k$ ;  $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and  $\epsilon^{\mu\nu\rho}$  is a totally-antisymmetric tensor with  $\epsilon^{012} = 1$ .

The electromagnetic coupling projected to the lowest Landau level is now written as

$$\bar{H}^{\text{em}} = \sum_{\mathbf{p}} \left\{ A_0[\mathbf{p}, t] + \frac{1}{2M} A_{12}[\mathbf{p}, t] + \mathcal{U}_2[\mathbf{p}, t] \right\} \rho_G^{(00)}[-\mathbf{p}, t], \quad (2.22)$$

$$\rho_G^{(00)}[\mathbf{p}, t] \equiv \int dy_0 \psi_0^{G\dagger}(y_0, t) e^{-\frac{1}{4}\mathbf{p}^2} e^{-i\mathbf{p} \cdot \mathbf{r}} \psi_0^G(y_0, t). \quad (2.23)$$

This  $\rho_G^{(00)}[\mathbf{p}, t]$  is the basic charge operator we use. It differs slightly from the charge operator projected to the lowest Landau level. Indeed, expressing  $\rho[\mathbf{p}, t] = \int d^2 x e^{-i\mathbf{p} \cdot \mathbf{x}} \psi^\dagger(x) \psi(x)$  in terms of  $\psi_n^G$  yields

$$\begin{aligned} \rho[\mathbf{p}, t] &= \int dy_0 \psi_n^{G\dagger}(y_0, t) W_{nn'} \psi_{n'}^G(y_0, t), \\ W_{nn'} &= G_{nm} F_{mm'}(-\mathbf{p}) e^{-\frac{1}{4}\mathbf{p}^2} e^{-i\mathbf{p} \cdot \mathbf{r}} (G^{-1})_{m'n'}, \end{aligned} \quad (2.24)$$

in obvious notation;  $F_{mm'}(-\mathbf{p})$  stands for the matrix (2.13). Let us extract the lowest-Landau-level component  $\int dy_0 \psi_0^{G\dagger} W_{00} \psi_0^G$  and denote it as  $\rho^{(00)}[\mathbf{p}, t] = \rho_G^{(00)}[\mathbf{p}, t] + \Delta \rho_G^{(00)}[\mathbf{p}, t]$ . The deviation to  $O(U)$  reads

$$\Delta \rho_G^{(00)}[\mathbf{p}, t] = \sum_{\mathbf{k}} \rho_G^{(00)}[\mathbf{p} - \mathbf{k}, t] u[\mathbf{p}, \mathbf{k}, t], \quad (2.25)$$

with

$$\begin{aligned} u[\mathbf{p}, \mathbf{k}, t] &= i e^{\frac{1}{4}\mathbf{k}^2} \sum_{s=1} \frac{1}{s!} \left\{ e^{-\frac{1}{2}\mathbf{k}^\dagger \cdot \mathbf{p}} \left( \frac{-i\mathbf{p}}{\sqrt{2}} \right)^s \eta_{0s}[\mathbf{k}, t] \right. \\ &\quad \left. - e^{-\frac{1}{2}\mathbf{p}^\dagger \cdot \mathbf{k}} \left( \frac{-i\mathbf{p}^\dagger}{\sqrt{2}} \right)^s \eta_{s0}[\mathbf{k}, t] \right\} \\ &= i p_j \left\{ \epsilon^{jk} A_k[\mathbf{k}, t] + O(A_{i0}) \right\} + O(\mathbf{k}). \end{aligned} \quad (2.26)$$

Here we have retained only terms with no derivatives  $\mathbf{k}$  or  $\partial_t$  acting on  $A_\mu[\mathbf{k}, t]$ , the portion needed later. As seen from Eq. (2.24),  $\rho[\mathbf{p}, t]$  is not invariant under  $W_\infty$  transformations  $G$ , except for the total charge  $\rho[\mathbf{p} = 0, t]$ . This explains why  $\rho^{(00)}[\mathbf{p}, t]$  differs from  $\rho_G^{(00)}[\mathbf{p}, t]$ .

Let us next consider the Coulomb interaction, which is a functional of the density  $\rho(x) = \psi^\dagger(x) \psi(x)$  or the deviation  $\delta \rho(x) = \rho(x) - \rho_{\text{av}}$  from the average electron density  $\rho_{\text{av}}$ . Passing to the Fourier space and setting  $\rho[\mathbf{p}, t] \rightarrow \rho^{(00)}[\mathbf{p}, t]$  yields the Coulomb interaction projected to the lowest Landau level

$$\bar{H}^{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} V[\mathbf{p}] \delta \bar{\rho}_{-\mathbf{p}} \delta \bar{\rho}_{\mathbf{p}} + \Delta H^{\text{Coul}}, \quad (2.27)$$

$$\Delta H^{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} V[\mathbf{p}] \sum_{\mathbf{k}} u[\mathbf{p}, \mathbf{k}, t] \{ \bar{\rho}_{-\mathbf{p}}, \bar{\rho}_{\mathbf{p}-\mathbf{k}} \}, \quad (2.28)$$

where we have denoted  $\bar{\rho}_{\mathbf{p}} \equiv \rho_G^{(00)}[\mathbf{p}, t]$  and  $\delta \bar{\rho}_{\mathbf{p}} \equiv \bar{\rho}_{\mathbf{p}} - \rho_{\text{av}} (2\pi)^2 \delta^2(\mathbf{p})$ , with obvious  $t$  dependence suppressed;  $V[\mathbf{p}] = V[-\mathbf{p}]$  denotes the Coulomb potential in momentum space. In Eq. (2.28) we have set  $\delta \bar{\rho}_{-\mathbf{p}} \rightarrow \bar{\rho}_{-\mathbf{p}}$  because the  $\rho_{\text{av}}$  term does not affect the electromagnetic response as long as  $V[\mathbf{0}] < \infty$  or even for  $V[\mathbf{p}] \sim 1/|\mathbf{p}|$ . The ( $W_\infty$ -breaking) correction  $\Delta H^{\text{Coul}}$  has an important consequence, as we shall see later.

The dynamics within the lowest Landau level is now governed by the Hamiltonian  $\bar{H} = \bar{H}^{\text{Coul}} + \bar{H}^{\text{em}}$ . [We have suppressed the quenched kinetic energy term  $\frac{1}{2} \omega_c$ .] Suppose now that an incompressible many-body state  $|G\rangle$  of uniform density  $\rho_{\text{av}} = \nu/(2\pi\ell^2)$  is formed within the lowest Landau level ( $\nu < 1$ ) via the Coulomb interaction (for  $A_\mu = 0$ ). Then its response to weak electromagnetic potentials  $A_\mu(x)$  is described by the rest of terms in  $\bar{H}$ . In particular, setting  $\langle G | \bar{\rho}_{-\mathbf{p}} | G \rangle = \rho_{\text{av}} (2\pi)^2 \delta^2(\mathbf{p})$  in  $\bar{H}^{\text{em}}$  one obtains the effective action to  $O(A^2)$ :

$$S^{\text{em}} = -\rho_{\text{av}} \int d^3 x [A_0(x) + \mathcal{U}_2(x)], \quad (2.29)$$

which is manifestly gauge invariant. The  $\mathcal{U}_2(x)$  summarizes the effect of electromagnetic inter-Landau-level mixing and agrees<sup>26</sup> with the result of a direct perturbative calculation<sup>9</sup> (for  $\nu = 1$ ). Note that this response is determined by the charge density  $\rho_{\text{av}}$  alone without knowing further details of the state  $|G\rangle$ .

The electromagnetic interaction in  $\bar{H}$  also gives rise to intra-Landau-level transitions. The inter-level cyclotron mode, however, saturates the oscillator-strength sum rule in accordance with Kohn's theorem<sup>27</sup> and, as a result, the intra-Landau-level excitations are only dipole-inactive<sup>23</sup> (i.e., the response vanishes faster than  $\mathbf{k}^2$  for  $\mathbf{k} \rightarrow 0$ ); this implies, in particular, that the quantum Hall states show universal  $O(\mathbf{k})$  and  $O(\mathbf{k}^2)$  long-wavelength electromagnetic characteristics determined by  $S^{\text{em}}$  above. The situation changes drastically for double-layer (or multi-layer) systems, which we discuss in the next section.

### III. DOUBLE-LAYER SYSTEMS

In this section we study the electromagnetic response of double-layer quantum Hall systems. Consider a double-layer system in the absence of interlayer tunneling, with average electron densities  $\rho_{\text{av}}^\alpha = (\rho_{\text{av}}^1, \rho_{\text{av}}^2)$  in the upper ( $\alpha = 1$ ) and lower ( $\alpha = 2$ ) layers. The electron fields  $\psi^\alpha$  in the two layers are taken to be coupled through the Coulomb interaction

$$H^{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} \delta\rho^\alpha[-\mathbf{p}, t] V_{\alpha\beta}[\mathbf{p}] \delta\rho^\beta[\mathbf{p}, t], \quad (3.1)$$

with  $\delta\rho^\alpha[\mathbf{p}, t] = \rho^\alpha[\mathbf{p}, t] - \rho_{\text{av}}^\alpha (2\pi)^2 \delta^2(\mathbf{p})$  and  $\rho^\alpha(x) = \psi^{\alpha\dagger}(x)\psi^\alpha(x)$ . Here  $V_{11}[\mathbf{p}] = V_{22}[\mathbf{p}]$  and  $V_{12}[\mathbf{p}] = V_{21}[\mathbf{p}]$  are the intralayer and interlayer potentials, respectively; summations over repeated layer-indices  $\alpha = 1, 2$  are understood from now on.

The system is placed in a common strong perpendicular magnetic field  $B$ . To probe each layer let us couple two weak fields  $A_\mu^\alpha = (A_\mu^1, A_\mu^2)$  to the two layers separately, and carry out projection onto the Landau levels for each layer. Then the dynamics within the lowest Landau level is governed by the Hamiltonian

$$\bar{H} = \frac{1}{2} \sum_{\mathbf{p}} V_{\alpha\beta}[\mathbf{p}] \delta\bar{\rho}_{-\mathbf{p}}^\alpha \delta\bar{\rho}_{\mathbf{p}}^\beta + \sum_{\mathbf{p}} \left( A_0^\alpha[\mathbf{p}] + \frac{1}{2M} A_{12}^\alpha[\mathbf{p}] \right) \bar{\rho}_{-\mathbf{p}}^\alpha + \Delta H^{\text{Coul}}, \quad (3.2)$$

where we have denoted  $\bar{\rho}_{\mathbf{p}}^\alpha \equiv \rho_G^{\alpha(00)}[\mathbf{p}, t]$ ,  $A_0^\alpha[\mathbf{p}, t] \rightarrow A_0^\alpha[\mathbf{p}]$ , etc., for short. [For conciseness we shall suppress such obvious time dependence in what follows; confusion may not arise since we mainly handle quantities at equal times or at a fixed time.] The response due to the cyclotron modes, including the  $\rho_{\text{av}}^\alpha A_0^\alpha$  coupling and isolated from  $\bar{H}$ , reads

$$S^{\text{cycl}} = - \int d^3x \rho_{\text{av}}^\alpha \left[ A_0^\alpha(x) + \mathcal{U}_2^\alpha(x) \right], \quad (3.3)$$

where  $\mathcal{U}_2^\alpha(x)$  stand for  $\mathcal{U}_2(x)$  in Eq. (2.20) with  $A_\mu \rightarrow A_\mu^\alpha$ .

It is convenient to use, instead of  $\bar{\rho}_{\mathbf{p}}^\alpha$ ,

$$\bar{\rho}_{\mathbf{p}} = \bar{\rho}_{\mathbf{p}}^1 + \bar{\rho}_{\mathbf{p}}^2, \quad \bar{d}_{\mathbf{p}} = \bar{\rho}_{\mathbf{p}}^1 - \bar{\rho}_{\mathbf{p}}^2, \quad (3.4)$$

and write the Coulomb interaction as

$$\bar{H}^{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} \left( V_{\mathbf{p}}^+ \delta\bar{\rho}_{-\mathbf{p}} \delta\bar{\rho}_{\mathbf{p}} + V_{\mathbf{p}}^- \delta\bar{d}_{-\mathbf{p}} \delta\bar{d}_{\mathbf{p}} \right) + \Delta H^{\text{Coul}}, \quad (3.5)$$

where

$$V_{\mathbf{p}}^\pm = \frac{1}{2} \left( V_{11}[\mathbf{p}] \pm V_{12}[\mathbf{p}] \right). \quad (3.6)$$

With an analogous  $O(2)$  rotation

$$A_\mu^1 = A_\mu^{\text{em}} + A_\mu^-, \quad A_\mu^2 = A_\mu^{\text{em}} - A_\mu^-, \quad (3.7)$$

$A_\mu^{\text{em}}$  is coupled to  $\bar{\rho}$  and  $A_\mu^-$  to  $\bar{d}$  in  $\bar{H}$ . Here  $A_\mu^{\text{em}}$  represents the electromagnetic potential that probes in-phase density fluctuations of the two layers while  $A_\mu^-$  probes the out-of-phase density fluctuations. The field-dependent Coulomb interaction term is now written as  $\Delta H^{\text{Coul}} = \Delta^+ H^{\text{Coul}} + \Delta^- H^{\text{Coul}}$ , with

$$\Delta^+ H^{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} V_{\mathbf{p}}^+ \sum_{\mathbf{k}} \left( u^{\text{em}}[\mathbf{p}, \mathbf{k}] \{ \bar{\rho}_{-\mathbf{p}}, \bar{\rho}_{\mathbf{p}-\mathbf{k}} \} + u^-[\mathbf{p}, \mathbf{k}] \{ \bar{\rho}_{-\mathbf{p}}, \bar{d}_{\mathbf{p}-\mathbf{k}} \} \right), \quad (3.8)$$

where  $u^{\text{em}}$  and  $u^-$  stand for  $u[\mathbf{p}, \mathbf{k}, t]$  in Eq. (2.26) with  $A_\mu$  replaced by  $A_\mu^{\text{em}}$  and  $A_\mu^-$ , respectively. For  $\Delta^- H^{\text{Coul}}$  simply replace  $V^+[\mathbf{p}] \rightarrow V^-[\mathbf{p}]$ ,  $\bar{\rho} \rightarrow \bar{d}$  and  $\bar{d} \rightarrow \bar{\rho}$  in the above.

We are now ready to discuss collective excitations within the lowest Landau level. The projected single-mode approximation<sup>23</sup> (SMA) provides a powerful means to study collective excitations of liquid states and, in particular, shows the presence of gapful density fluctuations in general single- and double-layer quantum Hall systems.<sup>16,18,23</sup>

Let  $|G\rangle$  denote the exact ground state of the double-layer system of our concern (for  $A_\mu = 0$ ). The SMA supposes that the density fluctuations over  $|G\rangle$  have predominant overlap with  $|G\rangle$  through the associated density operators  $\rho^\alpha$ . For the present system we consider two modes, a charge mode  $|\phi_{\mathbf{k}}^+\rangle \sim \bar{\rho}_{\mathbf{k}}|G\rangle$  representing the in-phase density fluctuations of the two layers and a phonon-roton mode  $|\phi_{\mathbf{k}}^-\rangle \sim \bar{d}_{\mathbf{k}}|G\rangle$  representing the inter-layer out-of-phase density fluctuations.

Let us normalize the  $|\phi_{\mathbf{k}}^-\rangle$  mode by writing

$$|\phi_{\mathbf{k}}^-\rangle = \frac{1}{\sqrt{2N \bar{s}^-(\mathbf{k})}} \bar{d}_{\mathbf{k}}|G\rangle \quad (3.9)$$

where the normalization of the wave function

$$\bar{s}^-(\mathbf{k}) \equiv \frac{1}{2N} \langle G | \bar{d}_{-\mathbf{k}} \bar{d}_{\mathbf{k}} | G \rangle = \bar{s}^-(-\mathbf{k}) \quad (3.10)$$

is nothing but the (projected) static structure factor of the ground state  $|G\rangle$ ;  $N = N^1 + N^2$  denotes the total

number of electrons. [To be precise,  $\bar{d}_{\mathbf{k}}$  in the above refers to  $\bar{d}_{\mathbf{k}}(t)$  at some fixed time, say,  $t = 0$ .] Similarly, we normalize  $|\phi_{\mathbf{k}}^{\pm}\rangle$  with  $\bar{s}^{\pm}(\mathbf{k}) = \langle G | \bar{\rho}_{-\mathbf{k}} \bar{\rho}_{\mathbf{k}} | G \rangle / (2N)$ .

In the SMA the static structure factors  $\bar{s}^{\pm}(\mathbf{k})$  are the basic quantities, through which the effect of non-trivial correlations pertinent to the ground state is reflected in the dynamics. Conservation laws reveal some of their general features:<sup>16,18</sup> Invariance under translations of both layers implies an analog of Kohn's theorem for the interlayer in-phase collective excitations so that  $\bar{s}^+(\mathbf{k}) \sim |\mathbf{k}|^4$  for small  $\mathbf{k}$ . As a result, the  $O(\mathbf{k})$  and  $O(\mathbf{k}^2)$  in-phase ( $A_{\mu}^{\text{em}}$ ) response of the double-layer system is governed by the cyclotron modes and is essentially the same as that of the single-layer system.

On the other hand, the presence of interlayer interactions  $V_{12}(\mathbf{p})$  spoils invariance under relative translations of the two layers and, unless spontaneous interlayer coherence is realized, the out-of-phase collective excitations become dipole-active,<sup>16,18</sup>

$$\bar{s}^-(\mathbf{k}) = c^- \frac{1}{2} \mathbf{k}^2 + O(|\mathbf{k}|^4). \quad (3.11)$$

With this fact in mind we shall henceforth concentrate on the dipole-active out-of-phase response. Actually, particular sets of double-layer FQH states we have in mind are the Halperin  $(m_1, m_2, n)$  or  $(m, m, n)$  states.<sup>13</sup> For the  $(m, m, n)$  states the coefficient  $c^-$  is explicitly known:<sup>16</sup>

$$c^- = \frac{n}{m-n}. \quad (3.12)$$

In this paper we do not discuss the case of the  $(m, m, m)$  states, where interlayer coherence develops<sup>20,21</sup> and which thus requires a separate analysis.

In the SMA the excitation energy  $\epsilon_{\mathbf{k}}^-$  of the collective mode is determined from the (projected) oscillator strength

$$\bar{f}^-(\mathbf{k}) = \frac{1}{2N} \langle G | \bar{d}_{-\mathbf{k}} (H - E_0) \bar{d}_{\mathbf{k}} | G \rangle \quad (3.13)$$

(with  $E_0$  being the ground state energy) by saturating it with the single mode  $|\phi_{\mathbf{k}}^-\rangle$ , so that

$$\epsilon_{\mathbf{k}}^- = \bar{f}^-(\mathbf{k}) / \bar{s}^-(\mathbf{k}). \quad (3.14)$$

This  $\bar{f}^-(\mathbf{k})$  is directly calculated (by first rewriting it as a double commutator  $\frac{1}{4N} \langle G | [\bar{d}_{-\mathbf{k}}, [H, \bar{d}_{\mathbf{k}}]] | G \rangle$  and) using the  $W_{\infty}$  algebra of the projected charges,<sup>23</sup>

$$\begin{aligned} [\bar{\rho}_{\mathbf{p}}, \bar{\rho}_{\mathbf{k}}] &= [\bar{d}_{\mathbf{p}}, \bar{d}_{\mathbf{k}}] = (e^{\frac{1}{2}p^{\dagger}k} - e^{\frac{1}{2}k^{\dagger}p}) \bar{\rho}_{\mathbf{p}+\mathbf{k}}, \\ [\bar{\rho}_{\mathbf{p}}, \bar{d}_{\mathbf{k}}] &= (e^{\frac{1}{2}p^{\dagger}k} - e^{\frac{1}{2}k^{\dagger}p}) \bar{d}_{\mathbf{p}+\mathbf{k}}, \end{aligned} \quad (3.15)$$

with the result<sup>16,18</sup>

$$\bar{f}^-(\mathbf{k}) = -\frac{\mathbf{k}^2}{2} \sum_{\mathbf{p}} \mathbf{p}^2 V_{12}[\mathbf{p}] \bar{s}^{12}(\mathbf{p}) + O(|\mathbf{k}|^4), \quad (3.16)$$

where  $\bar{s}^{12}(\mathbf{p}) \equiv \langle G | \bar{\rho}_{-\mathbf{p}}^1 \bar{\rho}_{\mathbf{p}}^2 | G \rangle / N = \frac{1}{2} \{ \bar{s}^+(\mathbf{p}) - \bar{s}^-(\mathbf{p}) \}$ . This leads to the SMA excitation gap at  $\mathbf{k} \rightarrow 0$ :

$$\epsilon_0^- \equiv \epsilon_{\mathbf{k}=0}^- = \frac{1}{c^-} \sum_{\mathbf{p}} V_{12}[\mathbf{p}] \mathbf{p}^2 \{ -\bar{s}^{12}(\mathbf{p}) \}, \quad (3.17)$$

where  $1/c^- = (m-n)/n$  for the  $(m, m, n)$  states.

Let us now study the response of the FQH ground states, which involves excitation of collective modes. Consider first the  $A_0^-$  to  $A_0^-$  response, and write the associated density-density response function  $(-i) \langle G | T \bar{d}(x) \bar{d}(x') | G \rangle$  in spectral form in Fourier space

$$F[\omega, \mathbf{k}] = \sum_n \left\{ \frac{1}{\omega - \epsilon_n} \sigma^n(\mathbf{k}) - \frac{1}{\omega + \epsilon_n} \sigma^n(-\mathbf{k}) \right\}, \quad (3.18)$$

$$\sigma^n(\mathbf{k}) = \frac{1}{\Omega} \langle G | \bar{d}_{\mathbf{k}} | n \rangle \langle n | \bar{d}_{-\mathbf{k}} | G \rangle, \quad (3.19)$$

where  $\epsilon_n \equiv E_n - E_0$  stands for the excitation energy of the intermediate state  $|n\rangle$ ;  $\Omega$  denotes the spatial surface area. In the SMA we saturate the sum over  $|n\rangle$  by a single collective mode  $|\phi_{\mathbf{k}}^-\rangle$  so that  $\sigma^n(\mathbf{k}) \rightarrow (2N/\Omega) \bar{s}^-(\mathbf{k})$ , and find, noting  $\sigma^n(\mathbf{k}) = \sigma^n(-\mathbf{k})$ ,

$$F[\omega, \mathbf{k}] = \rho_{\text{av}} \frac{4\epsilon_{\mathbf{k}}^-}{\omega^2 - (\epsilon_{\mathbf{k}}^-)^2} \bar{s}^-(\mathbf{k}), \quad (3.20)$$

where  $\rho_{\text{av}} = N/\Omega = \rho_{\text{av}}^1 + \rho_{\text{av}}^2$ . With  $\bar{s}^-(\mathbf{k}) \approx \frac{1}{2} c^- \mathbf{k}^2$ , this leads to a dipole response of the form

$$S_{A_0}^{\text{col}} = \frac{1}{2} \rho_{\text{av}} \int d^3x \partial_j A_0^-(x) \frac{2c^- \epsilon_0^-}{(\epsilon_0^-)^2 - \omega^2} \partial_j A_0^-(x), \quad (3.21)$$

where  $\omega$  stands for  $i\partial_t$ . In view of the simple structure of  $A_{\mu}^- \bar{d}$  coupling in  $\bar{H}$  of Eq. (3.2) it may appear somewhat disturbing that this response is not gauge invariant by itself. Fortunately, the field-dependent Coulomb interaction  $\Delta H^{\text{Coul}}$  serves to promote it into a gauge-invariant form and at the same time yields another important response, the Chern-Simons (CS) term, as shown below.

Let us consider the  $\Delta H^{\text{Coul}}$  to  $A_0^-$  response. It contains correlation functions like  $\langle G | T d_{\mathbf{k}}(t) \bar{I}_{\mathbf{p},\mathbf{k}}(0) | G \rangle$ , which we write in spectral form and saturate with a single collective mode  $|\phi_{\mathbf{k}}^-\rangle$  again, where

$$\bar{I}_{\mathbf{p},\mathbf{k}} = \{ \bar{\rho}_{-\mathbf{p}}, \bar{d}_{\mathbf{p}-\mathbf{k}} \}. \quad (3.22)$$

The result is

$$S_{\Delta H}^{\text{col}} = - \int dt \sum_{\mathbf{k}} A_0^-[-\mathbf{k}] \left\{ \frac{1}{\omega - \epsilon_{\mathbf{k}}^-} \sigma_{dI}^{\phi}(\mathbf{k}) - \frac{1}{\omega + \epsilon_{\mathbf{k}}^-} \sigma_{Id}^{\phi}(-\mathbf{k}) \right\}, \quad (3.23)$$

with

$$\sigma_{dI}^{\phi}(\mathbf{k}) = \frac{1}{2\Omega} \sum_{\mathbf{p}} u^-[\mathbf{p}, \mathbf{k}] \left\{ V_{\mathbf{p}}^+ g(\mathbf{p}, \mathbf{k}) + V_{\mathbf{p}}^- g(\mathbf{k}-\mathbf{p}, \mathbf{k}) \right\}, \quad (3.24)$$

$$g(\mathbf{p}, \mathbf{k}) = \langle G | \bar{d}_{\mathbf{k}} \bar{I}_{\mathbf{p},\mathbf{k}} | G \rangle. \quad (3.25)$$

For  $\sigma_{Id}^\phi(-\mathbf{k})$  replace  $g(\mathbf{p}, \mathbf{k})$  in Eq. (3.24) by  $\hat{g}(\mathbf{p}, \mathbf{k}) \equiv \langle G | \bar{I}_{\mathbf{p}, \mathbf{k}} \bar{d}_{\mathbf{k}} | G \rangle$  which equals  $g(-\mathbf{p}, -\mathbf{k})^\dagger$ .

It is now necessary to calculate the matrix elements of products of three charges or the three-particle distribution in the ground state  $|G\rangle$ . A similar problem was discussed earlier by Feenberg<sup>28</sup> in his attempt at improving the Bijl-Feynman theory of the roton spectrum of liquid  $^4\text{He}$ . His result adapted to the present case reads

$$\langle G | \bar{d}_{\mathbf{k}} \{ \bar{\rho}_{-\mathbf{p}}, \bar{d}_{\mathbf{p}-\mathbf{k}} \} | G \rangle \propto N \bar{s}^-(\mathbf{k}) \bar{s}^+(\mathbf{p}) \bar{s}^-(\mathbf{p}-\mathbf{k}) \quad (3.26)$$

provided the charges are mutually commuting like their unprojected counterparts (or more precisely, the arguments  $\mathbf{r} = (r_1, r_2)$  of the projected charges are taken to commute). If Eq. (3.26) were the case,  $g(\mathbf{p}, \mathbf{k})$  in  $\sigma_{dI}^\phi(\mathbf{k})$  would be of  $O(\mathbf{k}^2)$ , leading to responses involving three derivatives or more, i.e., no  $O(\mathbf{k}^2)$  response. This is the key observation and suggests that the desired  $O(\mathbf{k}^2)$  response is determined by isolating from the products of three charges the portion that derives from the noncommutative nature of the projected charges. Fortunately, for the  $O(\mathbf{k}^2)$  response it suffices to calculate  $g(\mathbf{p}, \mathbf{k})$  to first order in  $\mathbf{k}$ ; this is easily seen from the structure of  $u^-[ \mathbf{p}, \mathbf{k} ]$  [that the  $p^n$  terms are of  $O(k^{n-1})$ ] and by use of symmetric integration over  $\mathbf{p}$ .

For actual calculation we decompose the three-charge products into sums of normal-ordered products with respect to  $\psi_0^{G\dagger}$  and  $\psi_0^G$ , using the multiplication laws like  $\bar{\rho}_{\mathbf{p}} \bar{\rho}_{\mathbf{k}} = e^{\frac{1}{2} \mathbf{p}^\dagger \mathbf{k}} \bar{\rho}_{\mathbf{p}+\mathbf{k}} : \bar{\rho}_{\mathbf{p}} \bar{\rho}_{\mathbf{k}} :$ . The relevant noncommutative portion is thereby readily identified and is seen to consist of products of two charges at most; see Appendix A. In particular, the noncommutative portion of  $\bar{d}_{\mathbf{k}} \bar{I}_{\mathbf{p}, \mathbf{k}}$  has the structure

$$(\bar{d}_{\mathbf{k}} \bar{I}_{\mathbf{p}, \mathbf{k}})^{\text{NC}} = (i\mathbf{p} \times \mathbf{k} + \mathbf{p} \cdot \mathbf{k}) (\bar{\rho}_{-\mathbf{p}} \bar{\rho}_{\mathbf{p}} - \bar{d}_{\mathbf{k}-\mathbf{p}} \bar{d}_{\mathbf{p}-\mathbf{k}}) + O(k^0) \bar{d}_{-\mathbf{k}} \bar{d}_{\mathbf{k}} + O(k^2), \quad (3.27)$$

which leads to

$$\begin{aligned} \sigma_{dI}^\phi(\mathbf{k}) &= \rho_{\text{av}} \sum_{\mathbf{p}} u^-[ \mathbf{p}, \mathbf{k} ] \left\{ 2(i\mathbf{p} \times \mathbf{k} + \mathbf{p} \cdot \mathbf{k}) V_{12}[\mathbf{p}] \bar{s}^{12}(\mathbf{p}) + \dots \right\} \\ &= \rho_{\text{av}} c^- \epsilon_0^- (\delta^{jm} - i\epsilon^{jm}) k_j \left\{ A_m[\mathbf{k}] + O(A_{i0}) \right\} + O(k^3) \end{aligned} \quad (3.28)$$

where we have used Eqs. (2.26) and (3.17) to arrive at the last line. Similarly, for  $\sigma_{Id}^\phi(-\mathbf{k})$  we obtain the same expression (3.28) with  $\delta^{jm} \rightarrow -\delta^{jm}$ . [Here we remark that the difference  $\sigma_{dI}^\phi(\mathbf{k}) - \sigma_{Id}^\phi(-\mathbf{k})$  is directly calculated by use of the charge algebra (3.15); this offers an independent check of the calculation given above.] The  $O(A_{i0})$  terms in Eq. (3.28) give rise to nonleading corrections to the dipole term (3.21) [smaller by factor of  $O(\epsilon_0^-/\omega_c)$  or  $O(\partial_t/\omega_c)$ ] and are omitted here. Eventually one is led to a response of the form

$$S_{\Delta H}^{\text{col}} = \rho_{\text{av}} \int d^3x A_0^- \frac{2c^- \epsilon_0^-}{(\epsilon_0^-)^2 - \omega^2} \left[ \partial_0 \partial_j A_j^- - \epsilon_0^- \epsilon^{ij} \partial_i A_j^- \right]. \quad (3.29)$$

Likewise, the  $\langle G | T \Delta H^{\text{Coul}} \Delta H^{\text{Coul}} | G \rangle$  response function gives rise to the  $\partial_0 A_j^- (\dots) \partial_0 A_j^-$  and  $A_i^- (\dots) \epsilon^{ij} \partial_0 A_j^-$  terms that precisely combine with Eqs. (3.21) and (3.29) to form gauge-invariant expressions, as shown in Appendix B. Furthermore, substituting  $A_0^- \rightarrow A_0^- + (1/2M) A_{12}^-$  in Eq. (3.29) generates a gauge-invariant  $A_{12}^- (\dots) A_{12}^-$  term.

Finally, collecting terms so far obtained and adding also the cyclotron-mode contribution in Eq. (3.3) yields the complete dipole and related responses in the SMA,

$$\begin{aligned} S_{\text{eff}}^- &= \frac{\rho_{\text{av}}}{2} \int d^3x A_{j0}^- \left[ \frac{2c^- \epsilon_0^-}{(\epsilon_0^-)^2 - \omega^2} + \frac{\omega_c}{\omega_c^2 - \omega^2} \right] A_{j0}^- \\ &\quad - \frac{\rho_{\text{av}}}{2} \int d^3x A_\mu^- \left[ \frac{2c^- (\epsilon_0^-)^2}{(\epsilon_0^-)^2 - \omega^2} + \frac{\omega_c^2}{\omega_c^2 - \omega^2} \right] \epsilon^{\mu\nu\rho} \partial_\nu A_\rho^- \\ &\quad - \frac{\rho_{\text{av}}}{2M} \int d^3x A_{12}^- \left[ \frac{2c^- (\epsilon_0^-)^2}{(\epsilon_0^-)^2 - \omega^2} + \frac{\omega_c^2}{\omega_c^2 - \omega^2} \right] A_{12}^-. \end{aligned} \quad (3.30)$$

In the low-energy ( $\omega \ll \epsilon_0^-$ ) regime  $S_{\text{eff}}^-$  reads

$$\begin{aligned} S_{\text{eff}}^- &\approx \frac{1}{2} \rho_{\text{av}} \int d^3x \left[ \frac{2c^-}{\epsilon_0^-} (A_{j0}^-)^2 \right. \\ &\quad \left. - (2c^- + 1) \left\{ A_\mu^- \epsilon^{\mu\nu\rho} \partial_\nu A_\rho^- + \frac{1}{M} (A_{12}^-)^2 \right\} \right]. \end{aligned} \quad (3.31)$$

The effect of the collective mode is significant here. The dipole response acquires the scale  $\sim \epsilon_0^-/(2c^-)$ , and the Hall conductance and magnetic susceptibility are enhanced by common factor  $(2c^- + 1)$ , which equals  $(m+n)/(m-n)$  for the  $(m, m, n)$  states. Here we see that the static structure factor of the  $(m, m, n)$  states precisely reproduces the effect of statistical-flux attachment employed in the CS theories: The Hall conductance  $\sigma_{xy}^- = -e^2 \rho_{\text{av}} (m+n)/(m-n)$  is exactly what one gets in the CS theories<sup>21,22</sup>, where, however, the collective-excitation spectrum is put on the scale of  $\omega_c$ .

Note here that the static density correlation functions are not sensitive to such a scale change.<sup>18</sup> They are given by  $\rho_{\text{av}} (2c^- + 1) \mathbf{p}^2 = \mathbf{p}^2 / \{\pi(m-n)\}$  for  $\langle G | d_{-\mathbf{p}} d_{\mathbf{p}} | G \rangle$  and  $\rho_{\text{av}} \mathbf{p}^2 = \mathbf{p}^2 / \{\pi(m+n)\}$  for  $\langle G | \rho_{-\mathbf{p}} \rho_{\mathbf{p}} | G \rangle$  in the long wavelength limit. In this way, the cyclotron mode and the collective mode contribute to the (unprojected) static structure factor in comparable magnitude, i.e., 1 to  $2c^-$  in ratio so that for the oscillator strength the collective-mode contribution is smaller by factor  $2c^- (\epsilon_0^-/\omega_c) < 1$ .

It will perhaps be worth remarking here that, while the SMA is a variational method in principle, it has led to an electromagnetic response of gauge-invariant form. This is consistent with an earlier observation<sup>23</sup> that the SMA is better suited for quantum Hall systems than the case of helium where the backflow corrections were necessary for restoring gauge invariance.

#### IV. EFFECTIVE GAUGE THEORY

In the previous section we have calculated the response of a double-layer electron system. Once such a response is known, it is possible to reconstruct it through the quantum fluctuations of a boson field. This procedure, known as functional bosonization,<sup>29</sup> was previously applied to single-layer systems to derive<sup>11</sup> an effective theory of the FQHE. In this section we construct an effective theory for the double-layer system.

For bosonization we take a shortcut and quote only a general formula. Consider a three-vector field  $b_\mu$  coupled to an external field  $A_\mu$ , with a Lagrangian of the form

$$L[b] = -A_\mu \epsilon^{\mu\nu\rho} \partial_\nu b_\rho + \frac{1}{2} b_\mu \epsilon^{\mu\nu\rho} \beta \partial_\nu b_\rho + \frac{1}{2} b_{k0} \frac{\beta}{\alpha} b_{k0} - \frac{1}{2} b_{12} \beta \sigma b_{12} - \kappa b_0, \quad (4.1)$$

where  $\beta$ ,  $\alpha$  and  $\sigma$  may contain derivatives and  $\kappa$  is a real constant;  $b_{\mu\nu} \equiv \partial_\mu b_\nu - \partial_\nu b_\mu$  and  $\epsilon^{012} = 1$ . Direct functional integration over  $b_\mu$  (with suitable gauge fixing) shows that this vector-field theory leads to an effective Lagrangian of the form<sup>11</sup>

$$L_{\text{eff}}[A] = -\frac{1}{2} A_\mu \frac{\alpha^2}{\beta \mathcal{D}} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{1}{2} A_{k0} \frac{\alpha}{\beta \mathcal{D}} A_{k0} - \frac{1}{2} A_{12} \frac{\sigma \alpha^2}{\beta \mathcal{D}} A_{12} - \frac{\kappa}{\beta} A_0 \quad (4.2)$$

in obvious compact notation, where

$$\mathcal{D} = \alpha^2 + \partial_t^2 - \alpha \sigma \nabla^2. \quad (4.3)$$

With this formula, it is straightforward to verify that, of the out-of-phase response  $S_{\text{eff}}^-$  in Eq. (3.30), the collective-mode contribution is reconstructed from the theory of a vector field  $\xi_\mu$ , with the Lagrangian

$$L_\xi^{\text{col}} = -A_\mu^- \epsilon^{\mu\nu\rho} \partial_\nu \xi_\rho + \frac{1}{4c^- \rho_{\text{av}}} \left[ \xi_\mu \epsilon^{\mu\nu\rho} \partial_\nu \xi_\rho + \frac{1}{\epsilon_0} (\xi_{k0})^2 - \frac{1}{M} (\xi_{12})^2 \right]. \quad (4.4)$$

Similarly, the cyclotron-mode contribution  $S^{\text{cycl}}$  of Eq. (3.3), to the dipole and related order, is reproduced from an analogous Lagrangian consisting of a pair of vector fields  $b_\mu^\alpha = (b_\mu^1, b_\mu^2)$ ,

$$L_b^{\text{cycl}} = -A^\alpha \epsilon \partial b^\alpha - b_0^\alpha + \frac{1}{2\rho_{\text{av}}^\alpha} \left[ b^\alpha \epsilon \partial b^\alpha + \frac{1}{\omega_c} (b_{k0}^\alpha)^2 - \frac{1}{M} (b_{12}^\alpha)^2 \right], \quad (4.5)$$

where summations over layer indices  $\alpha$  are understood;  $A^\alpha \epsilon \partial b^\alpha \equiv A_\mu^\alpha \epsilon^{\mu\nu\rho} \partial_\nu b_\rho^\alpha$ , etc., for short. Thus the double-layer system is now described by a gauge-field theory with the simple Lagrangian  $L_\xi^{\text{col}} + L_b^{\text{cycl}}$ .

The Lagrangian  $L_b^{\text{cycl}}$  reminds us of the dual-field Lagrangian of Lee and Zhang<sup>6</sup> (LZ), derived within the

Chern-Simons-Landau-Ginzburg theory and describing the long-wavelength characteristics of the FQHE. The LZ Lagrangian, generalized to describe the  $(m_1, m_2, n)$  Halperin states of the double-layer system, is written in terms of a pair of vector fields  $b_\mu^\alpha = (b_\mu^1, b_\mu^2)$ :

$$L_{\text{eff}}^{\text{CS}}[b] = -A^\alpha \epsilon \partial b^\alpha - b_0^\alpha + \frac{1}{2} \Lambda_{\alpha\beta} b^\alpha \epsilon \partial b^\beta + \frac{1}{2} \frac{1}{\omega_c \rho_{\text{av}}^\alpha} (b_{k0}^\alpha)^2 + \dots \quad (4.6)$$

The omitted terms ( $\dots$ ) contain Coulomb interactions and higher derivative terms, not relevant to our present discussion. The mixing matrix

$$\Lambda_{\alpha\beta} = 2\pi \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \quad (4.7)$$

is a consequence of statistical-flux attachment characteristic of the  $(m_1, m_2, n)$  states.

The LZ Lagrangian (4.6) rests on the composite-boson picture of the FQHE. Actually it is possible<sup>11</sup> to derive it as an effective theory reconstructed from the response of the composite fermions. Thus  $L_{\text{eff}}^{\text{CS}}[b]$  is a consequence of the CS theories, both bosonic and fermionic.<sup>30</sup>

Our effective Lagrangian  $L_\xi^{\text{col}} + L_b^{\text{cycl}}$  consists of three vector fields  $(\xi_\mu, b_\mu^\alpha)$  representing one collective mode and two cyclotron modes. To make a connection with  $L_{\text{eff}}^{\text{CS}}[b]$  let us try to rewrite  $L_\xi^{\text{col}} + L_b^{\text{cycl}}$  in favor of the new fields  $\eta_\mu^1 = b_\mu^1 + \frac{1}{2}\xi_\mu$  and  $\eta_\mu^2 = b_\mu^2 - \frac{1}{2}\xi_\mu$ , eliminate the collective mode  $\xi_\mu$  and retain terms up to  $O(\nabla^2)$ . The result is

$$L_{\text{eff}}[\eta] = -A^\alpha \epsilon \partial \eta^\alpha - \eta_0^\alpha + \frac{1}{2} \hat{\Lambda}_{\alpha\beta} \eta^\alpha \epsilon \partial \eta^\beta + \frac{1}{2} \left[ \frac{1}{\omega_c} \hat{\Lambda}_{\alpha\beta} + \left( \frac{1}{\epsilon_0} - \frac{1}{\omega_c} \right) \Xi_{\alpha\beta} \right] \eta_{k0}^\alpha \eta_{k0}^\beta - \frac{1}{2M} \hat{\Lambda}_{\alpha\beta} \eta_{12}^\alpha \eta_{12}^\beta + \dots, \quad (4.8)$$

with the mixing matrices

$$\hat{\Lambda}_{\alpha\beta} = 2\pi\lambda \begin{pmatrix} 1 + (2/c^-)\nu_2/\nu & 1 \\ 1 & 1 + (2/c^-)\nu_1/\nu \end{pmatrix}, \quad (4.9)$$

$$\Xi_{\alpha\beta} = 2\pi\lambda\Xi \begin{pmatrix} \nu_2/\nu_1 & -1 \\ -1 & \nu_1/\nu_2 \end{pmatrix}, \quad (4.10)$$

$$\lambda = 1/\left(\nu + \frac{2}{c^-} \frac{\nu_1\nu_2}{\nu}\right), \quad \lambda\Xi = \frac{2}{c^-} \frac{\nu_1\nu_2}{\nu} \lambda^2, \quad (4.11)$$

where  $\nu_\alpha = 2\pi\ell^2\rho_{\text{av}}^\alpha$  denote the filling factors for each layer and  $\nu = \nu_1 + \nu_2$ . Note that  $\Xi_{\alpha\beta}$  is essentially a projection operator.

Let us here take as the ground state  $|G\rangle$  the  $(m_1, m_2, n)$  state, for which  $\nu_1 = (m_2 - n)/(m_1 m_2 - n^2)$  and  $\nu_2 = (m_1 - n)/(m_1 m_2 - n^2)$ . Try to ask if  $L_{\text{eff}}[\eta]$  and  $L_{\text{eff}}^{\text{CS}}[b]$  could share the same  $O(\partial)$  long-wavelength structure or the CS term, i.e.,  $\hat{\Lambda}_{\alpha\beta} = \Lambda_{\alpha\beta}$ . The answer is affirmative. This fixes  $c^-$  uniquely,

$$c^- = \frac{2n}{m_1 + m_2 - 2n}, \quad (4.12)$$

and yields  $\lambda = n$  and  $\lambda_{\Xi} = n(1 - n\nu)$ . We have thus determined what appears to be the static structure factor to  $O(\mathbf{k}^2)$  of the  $(m_1, m_2, n)$  state, though not calculated directly so far; it is correctly reduced to the known value (3.12) for the  $(m, m, n)$  state.

While  $L_{\text{eff}}[\eta]$  and  $L_{\text{eff}}^{\text{CS}}[b]$  coincide to  $O(\partial)$ , they differ in  $O(\partial^2)$ , especially in the out-of-phase collective-mode spectrum. Let us explore the difference in detail for the  $(m, m, n)$  states. The  $L_{\text{eff}}[\eta]$  and  $L_{\text{eff}}^{\text{CS}}[b]$  are split into two sectors with  $\eta_{\mu}^{\pm} = \eta_{\mu}^1 \pm \eta_{\mu}^2$  and  $b_{\mu}^{\pm} = b_{\mu}^1 \pm b_{\mu}^2$ , respectively. The  $\eta_{\mu}^+$  sector is given by the Lagrangian (4.1) with  $b_{\mu} \rightarrow \eta_{\mu}^+$ ,  $A_{\mu} \rightarrow A_{\mu}^{\text{em}}$ ,  $\kappa \rightarrow 1$ ,  $\beta \rightarrow 1/\rho_{\text{av}} = \pi(m+n)$  and  $\alpha \rightarrow \omega_c$ , in precise agreement with the  $b_{\mu}^+$  sector of  $L_{\text{eff}}^{\text{CS}}[b]$ . On the other hand, the  $\eta_{\mu}^-$  sector is given by the Lagrangian (4.1) with  $b_{\mu} \rightarrow \eta_{\mu}^-$ ,  $A_{\mu} \rightarrow A_{\mu}^-$ ,  $\kappa \rightarrow 0$  and

$$\beta \rightarrow \pi(m-n), \quad \frac{1}{\alpha} \rightarrow \frac{1}{\epsilon_0^-} \frac{2n}{m+n} + \frac{1}{\omega_c} \frac{m-n}{m+n}. \quad (4.13)$$

The  $b_{\mu}^-$  sector of  $L_{\text{eff}}^{\text{CS}}[b]$  differs merely by the collective-mode spectrum

$$\alpha \rightarrow \alpha_{\text{CS}}^- = \pi(m-n) \rho_{\text{av}} \omega_c = \frac{m-n}{m+n} \omega_c. \quad (4.14)$$

The static density correlation functions  $\sim (1/\beta)\mathbf{p}^2$  are read from these  $\beta$  values  $\pi(m \pm n)$  and agree in both theories, as noted in Eq. (3.30). The  $1/\alpha$  in Eq. (4.13) correctly recovers the energy scale in the  $\omega \rightarrow 0$  limit of Eq. (3.30); here the collective-mode energy is apparently shifted to  $\frac{m+n}{2n} \epsilon_0^-$  from the on-resonance value  $\epsilon_0^-$ .

From the above consideration emerge the following observations: The presence of the dipole-active out-of-phase collective excitations, inherent to double-layer systems in general and specifically to the  $(m_1, m_2, n)$  Halperin states, implies strong interlayer correlations that affect the  $O(\partial)$  and  $O(\partial^2)$  long-wavelength characteristics of the double-layer FQH states. The leading  $O(\partial)$  correlations are correctly incorporated into the CS theories by the flux attachment transformation, which, however, fails to take in the next-leading  $O(\partial^2)$  correlations at least in the random-phase approximation; as a result, the collective-excitation spectrum is left on the scale of  $O(\omega_c)$ . In view of this, a practical way to improve the CS theories would be to regard the collective-excitation energy as a parameter to be adjusted phenomenologically.

Finally we wish to discuss vortex excitations in double-layer systems. An incompressible FQH state supports vortex excitations.<sup>1</sup> Vortices are readily introduced into the bosonic effective theories by replacing  $A_{\mu}^{\alpha} \rightarrow A_{\mu}^{\alpha} + \partial_{\mu} \Theta^{\alpha}$ , where  $\Theta^{\alpha}(x)$  stands for a topologically nontrivial component of the phase of the electron field in each layer.<sup>6</sup> The  $L_{\text{eff}}^{\text{CS}}[b]$  and  $L_{\text{eff}}[\eta]$  thereby acquire a vortex coupling of the same form  $-2\pi \tilde{j}_{\mu}^{\alpha} b_{\mu}^{\alpha}$  and  $-2\pi \tilde{j}_{\mu}^{\alpha} \eta_{\mu}^{\alpha}$ , respectively, where  $\tilde{j}_{\mu}^{\alpha} = (1/2\pi) \epsilon^{\mu\nu\rho} \partial_{\nu} \partial_{\rho} \Theta^{\alpha}$  denotes the vortex three-current

$$\tilde{j}_{\mu}^{\alpha}(x) = \sum_i [1, \partial_t \mathbf{x}^{(i)}(t)] q_i^{\alpha} \delta^2(\mathbf{x} - \mathbf{x}^{(i)}(t)), \quad (4.15)$$

with  $\mathbf{x}^{(i)}(t)$  standing for the trajectory of the  $i$ th vortex of vorticity  $q_i^{\alpha} = \pm 1, \pm 2, \dots$  in layer  $\alpha$ . The vortex charges are easily read from these vortex couplings.<sup>11</sup> The electromagnetic coupling  $-A^{\alpha} \epsilon \partial b^{\alpha}$  induces in  $b_{\mu}^{\alpha}$  an  $A_{\mu}^{\alpha}$ -dependent piece which is isolated by writing  $b_{\mu}^{\alpha} = b'_{\mu}^{\alpha} + f_{\mu}^{\alpha}[A]$  and choosing  $f_{\mu}^{\alpha}[A] = (\Lambda^{-1})_{\alpha\beta} A_{\mu}^{\beta} + O(\partial A)$ . As a result, the vortex is coupled to  $A_{\mu}^{\alpha}$  through this background piece  $-2\pi \tilde{j}_{\mu}^{\alpha} b_{\mu}^{\alpha} = -2\pi \tilde{j}_{\mu}^{\alpha} (\Lambda^{-1})_{\alpha\beta} A_{\mu}^{\beta} + \dots$ . This reveals that a vortex of vorticity  $(q^1, q^2)$  in each layer induces the amount of charge  $Q_v^{\alpha} = -2\pi q_i^{\beta} (\Lambda^{-1})_{\beta\alpha}$  in the two layers, or explicitly,

$$Q_v^1 = -e \frac{m_2 q^1 - n q^2}{m_1 m_2 - n^2}, \quad Q_v^2 = -e \frac{-n q^1 + m_1 q^2}{m_1 m_2 - n^2} \quad (4.16)$$

for the  $(m_1 m_2, n)$  states, in agreement with earlier results.<sup>21</sup>

## V. SUMMARY AND DISCUSSION

In this paper we have studied the electromagnetic characteristics of double-layer quantum Hall systems and derived an effective gauge theory of the FQHE. Our approach has clarified, above all, that it is the dipole-active excitations, both elementary and collective, that govern the transport properties of quantum Hall systems. In particular, single-layer systems support no dipole-active intra-Landau-level collective excitations, and consequently the incompressible FQH states show universal long-wavelength electromagnetic characteristics, governed by the inter-level cyclotron mode alone. In contrast, for double-layer systems interlayer out-of-phase collective excitations become dipole-active and alter the response of the systems fundamentally. The effective theory constructed from the response via bosonization is written in terms of three vector fields which precisely reflect the three dipole-active modes, i.e., one out-of-phase collective mode and two cyclotron modes, and properly incorporates the single-mode-approximation spectrum of collective excitations.

The presence of the dipole-active interlayer collective excitations implies strong interlayer correlations that affect the  $O(\partial)$  and  $O(\partial^2)$  long-wavelength characteristics of the double-layer FQH states. The  $O(\partial)$  correlations are correctly taken care of by the flux attachment transformation in the Chern-Simons theories, both bosonic and fermionic. The flux attachment, however, fails to take in the next-leading  $O(\partial^2)$  correlations (at least in the random-phase approximation). This explains why the Chern-Simons theories properly account for  $O(\partial)$  features, like the Hall conductance, vortex charges and long-range orders, while they leave the collective-excitation spectrum on the scale of the Landau gap  $\sim \omega_c$ .



The bosonization approach presented here gives rise to an effective theory of the FQHE in a manner logically independent of the Chern-Simons theories. It by itself does not tell at which filling fractions the FQH states emerge. Instead, it tells us that the long-wavelength characteristics of the incompressible FQH states are determined independent of the composite-boson or composite-fermion picture. It has thus allowed one to derive, for double-layer systems, an effective theory that embodies the SMA spectrum of collective excitations.

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## APPENDIX A: REDUCTION OF THREE-CHARGE PRODUCTS

In this appendix we outline how to isolate from products of projected charges the portion that originates from the noncommutative nature of  $r_i$ . Consider a product of the form  $I = \bar{d}_{\mathbf{p}_1} \bar{\rho}_{\mathbf{p}_2} \bar{d}_{\mathbf{p}_3}$ . By repeated use of the multiplication laws like  $\bar{\rho}_{\mathbf{p}} \bar{d}_{\mathbf{k}} = e^{\frac{1}{2} p^\dagger k} \bar{d}_{\mathbf{p}+\mathbf{k}} + : \bar{\rho}_{\mathbf{p}} \bar{d}_{\mathbf{k}} :$  one can decompose  $I$  into normal-ordered products:

$$\begin{aligned} \bar{d}_{\mathbf{p}_1} \bar{\rho}_{\mathbf{p}_2} \bar{d}_{\mathbf{p}_3} &= e^{\frac{1}{2} [p_1^\dagger (p_2 + p_3) + p_2^\dagger p_3]} \bar{\rho}_{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3} \\ &+ e^{\frac{1}{2} p_2^\dagger p_3} : \bar{d}_{\mathbf{p}_1} \bar{d}_{\mathbf{p}_2 + \mathbf{p}_3} : + e^{\frac{1}{2} p_1^\dagger p_3} : \bar{\rho}_{\mathbf{p}_2} \bar{\rho}_{\mathbf{p}_1 + \mathbf{p}_3} : \\ &+ e^{\frac{1}{2} p_1^\dagger p_2} : \bar{d}_{\mathbf{p}_1 + \mathbf{p}_2} \bar{d}_{\mathbf{p}_3} : + : \bar{d}_{\mathbf{p}_1} \bar{\rho}_{\mathbf{p}_2} \bar{d}_{\mathbf{p}_3} : . \quad (\text{A1}) \end{aligned}$$

The momentum dependent coefficients all derive from the operator nature of  $r_i$ . Thus it is easy to identify the intrinsically noncommutative contribution

$$\begin{aligned} I^{\text{NC}} &= (e^{\frac{1}{2} p_2^\dagger p_3} - 1) \bar{d}_{\mathbf{p}_1} \bar{d}_{\mathbf{p}_2 + \mathbf{p}_3} + (e^{\frac{1}{2} p_1^\dagger p_3} - 1) \bar{\rho}_{\mathbf{p}_2} \bar{\rho}_{\mathbf{p}_1 + \mathbf{p}_3} \\ &+ (e^{\frac{1}{2} p_1^\dagger p_2} - 1) \bar{d}_{\mathbf{p}_1 + \mathbf{p}_2} \bar{d}_{\mathbf{p}_3} + C \bar{\rho}_{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3}, \quad (\text{A2}) \end{aligned}$$

where

$$\begin{aligned} C &= e^{\frac{1}{2} p_1^\dagger (p_2 + p_3)} - 1 - (e^{\frac{1}{2} p_1^\dagger p_3} - 1) e^{\frac{1}{2} p_2^\dagger (p_1 + p_3)} \\ &- (e^{\frac{1}{2} p_1^\dagger p_2} - 1) e^{\frac{1}{2} (p_1^\dagger + p_2^\dagger) p_3}. \quad (\text{A3}) \end{aligned}$$

It is now clear how to write down analogous formulas for general three-charge products.

## APPENDIX B: CALCULATION OF A RESPONSE

In this appendix we outline the calculation of the correlation function  $-i \langle G | T \Delta H^{\text{Coul}} \Delta H^{\text{Coul}} | G \rangle$  contributing to  $S_{\text{eff}}$  in Eq. (3.30). Let us write it in spectral form and saturate it with a single collective mode  $|\phi_{\mathbf{k}}^-\rangle$ . The spectral weight thereby takes a simple and suggestive form

$$\sigma_{\Delta H \Delta H}^{\phi}(\mathbf{k}) = \Omega \frac{1}{2N \bar{s}^-(\mathbf{k})} \sigma_{Id}^{\phi}(-\mathbf{k}) \sigma_{dI}^{\phi}(-\mathbf{k}), \quad (\text{B1})$$

with  $\sigma_{dI}^{\phi}(\mathbf{k})$  and  $\sigma_{Id}^{\phi}(\mathbf{k})$  defined in Eq. (3.24). On substitution of Eq. (3.28) and removing total divergences,  $\sigma_{Id}^{\phi}(-\mathbf{k}) \sigma_{dI}^{\phi}(-\mathbf{k})$  turns into  $\gamma A_i[-\mathbf{k}] \mathbf{k}^2 A_i[\mathbf{k}]$  with  $\gamma = (\rho_{\text{av}} c^- \epsilon_0^-)^2$  while  $\sigma_{Id}^{\phi}(-\mathbf{k}) \omega \sigma_{dI}^{\phi}(-\mathbf{k})$  yields  $-i \gamma \epsilon^{ij} A_i[-\mathbf{k}] \omega \mathbf{k}^2 A_j[\mathbf{k}]$ . These give rise to the  $\partial_0 A_j^- [(\epsilon_0^-)^2 - \omega^2]^{-1} \partial_0 A_j^-$  and  $\epsilon^{ij} A_i [(\epsilon_0^-)^2 - \omega^2]^{-1} \partial_0 A_j^-$  terms in Eq. (3.30).

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